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LETTER TO THE EDITOR

q-deformations of the O(3) symmetric spin-1 Heisenberg chain

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Abstract. We present the general expression for the spin-1 Heisenberg chain invariant under the $U_q[SO(3)]$ quantum algebra. Several physical and mathematical implications are discussed.

Recently there has been a considerable interest in the properties of the one-dimensional O(3) symmetric spin-1 Heisenberg chain defined by the Hamiltonian (Affleck *et al* 1988, Papanicolaou 1988, Barber and Batchelor 1989b and references therein):

$$H = \alpha \sum_{k=1}^{n} O_k(a, b)$$

$$O_k(a, b) = a(\boldsymbol{S}_k \cdot \boldsymbol{S}_{k+1}) + b(\boldsymbol{S}_k \cdot \boldsymbol{S}_{k+1})^2 \qquad (1)$$

where $\alpha = +1(-1)$ corresponds to the antiferromagnetic (ferromagnetic) chain ($a \geqslant 0$). The eigenvalues of the matrix O_k are

$$E_0 = -2a + 4b \qquad E_1 = -a + b \qquad E_2 = a + b. \qquad (2)$$

E_J is the eigenvalue corresponding to total angular momentum J (where $\boldsymbol{S}_k + \boldsymbol{S}_{k+1} = \boldsymbol{J}$). Exact results have been obtained when two of the three eigenvalues coincide. We thus know that for $a = b = \alpha = 1$ (Sutherland 1975, de Vega 1989) the system is massless with a central charge $c = 2$ of the Virasoro algebra. This point is SU(3) symmetric ($E_2 = E_0$) and corresponds to the $\mathbf{3} \times \mathbf{3} = \mathbf{6} + \bar{\mathbf{3}}$ branching rules. Furthermore, the matrices

$$V_k = O_k(1, 1) - 1 \qquad (3)$$

satisfy the Hecke algebra (Jimbo 1986):

$$V_k^2 = 1 \qquad [V_k, V_{k'}] = 0 \qquad (|k - k'| \geqslant 2) \qquad (4a)$$

$$V_k V_{k+1} V_k = V_{k+1} V_k V_{k+1}. \qquad (4b)$$

The point $a = 0$, $b = \alpha = 1 (E_2 = E_1)$ also has SU(3) invariance, corresponding to the $\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8}$ branching rules. The matrices

$$U_k = 0_k(0, 1) - 1 \qquad (5)$$

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satisfy the Temperley-Lieb algebra (Barber and Batchelor 1989a)

$$U_k^2 = \beta U_k \qquad [U_k, U_{k'}] = 0 \qquad (|k-k'| \geq 2) \tag{6a}$$

$$U_k U_{k+1} U_k = U_k \tag{6b}$$

with $\beta = 3$. We remind the reader that according to the lore derived from the spin-$\frac{1}{2}$ Heisenberg chain, a system has a second-order phase transition if $-2 \leq \beta \leq 2$ and a first-order transition otherwise. The fact that the Hamiltonian with $O_k(0, 1)$ describes a massive phase was confirmed by the inversion relation calculations of Klümper (1989a, b). The case $\alpha = a = 1$, $b = 1/3$ ($E_1 = E_0$) was solved by Affleck *et al* (1988), who obtained again a massive phase. Finally the case $a = -b = 1$ (no two E_Js have the same value) corresponds to the O(3) symmetric Zamolodchikov-Fateev (1980) chain. Here one obtains a massless phase with a central charge $c = 3/2$ for $\alpha = 1$ (Alcaraz and Martins 1989a and references therein) and $c = 1$ for $\alpha = -1$ (Alcaraz and Martins 1989b, Baranowski and Rittenberg 1990).

In the present letter we consider the q-deformation of the Hamiltonian (1). We first define the $U_q[\mathrm{SU}(2)]$ quantum algebra (see, e.g., Jimbo 1985):

$$[\bar{S}^z, \bar{S}^\pm] = \pm \bar{S}^\pm \qquad [\bar{S}^+, \bar{S}^-] = [2\bar{S}^z]_q \tag{7}$$

where

$$[x]_q = (q^x - q^{-x})(q - q^{-1})^{-1}. \tag{8}$$

For the three-dimensional representation ($J = 1$) we have

$$\bar{S}^\pm = \sqrt{\frac{[2]_q}{2}}\, S^\pm \qquad \bar{S}^z = S^z \tag{9}$$

where the $S^\pm$, S^z matrices are the same as those used in (1) ($S^\pm = S^x \pm \mathrm{i}S^y$). Using the Casimir operator

$$C_2 = \bar{S}^- \bar{S}^+ + ([\bar{S}^z + \tfrac{1}{2}]_q)^2 \tag{10}$$

one can construct $U_q[\mathrm{SU}[2]]$ invariant quantities. We consider the tensor product (co-multiplication) of two $J = 1$ representations:

$$\bar{S}^\pm = \bar{S}_k^\pm q^{\bar{S}_{k+1}^z} + q^{-\bar{S}_k^z} \bar{S}_{k+1}^\pm = \sqrt{\frac{[2]_q}{2}}\,(S_k^\pm q^{S_{k+1}^z} + q^{-S_k^z} S_{k+1}^\pm).$$

$$q^{\bar{S}^z} = q^{S_k^z} q^{S_{k+1}^z}. \tag{11}$$

Using (11) we can compute C_2 and $(C_2)^2$ in order to derive the q-deformation of $O_k(a, b)$, which we denote by $O_k(a, b; q)$. We obtain

$$\begin{aligned} O_k(a, b; q) = {} & a(\boldsymbol{S}_k \cdot \boldsymbol{S}_{k+1}) + b(\boldsymbol{S}_k \cdot \boldsymbol{S}_{k+1})^2 \\ & + (\sinh^2 \lambda)[2a(S_k^z)^2 + 2a(S_{k+1}^z)^2 + (a-b)(S_k^z S_{k+1}^z - (S_k^z)^2 (S_{k+1}^z)^2)] \\ & + \tfrac{1}{2}(a+b)(\sinh \lambda)[(S_k^x S_{k+1}^x + S_k^y S_{k+1}^y)(S_{k+1}^z - S_k^z) + \mathrm{HC}] \\ & + 2(b-a)\left(\sinh^2 \frac{\lambda}{2}\right)[(S_k^x S_{k+1}^x + S_k^y S_{k+1}^y) S_k^z S_{k+1}^z + \mathrm{HC}] \\ & + (\sinh 2\lambda)[a(S_{k+1}^z - S_k^z) + \tfrac{1}{2}(a+b) S_k^z S_{k+1}^z (S_{k+1}^z - S_k^z)] \end{aligned} \tag{12}$$

where $q = \mathrm{e}^\lambda$. The eigenvalues of $O_k(a, b; q)$, which are q-dependent, are

$$E_0(q) = -2a + b(q + q^{-1})^2 \qquad E_1(q) = -a + b$$

$$E_2(q) = a(q^2 + q^{-2} - 1) + b. \tag{13}$$

For $q=1$ we recover the values given in (2). We are now about to have some surprises. If you take $a=b=1$ we have $E_2(q)=E_0(q)$ as in the $q=1$ case. This result is unexpected since the $U_q[\mathrm{SO}(3)]$ should break the SU(3) multiplet structure. At the same time, the $U_q[\mathrm{SU}(3)]$ invariant Hamiltonian obtained by taking the $\mathbf{3}\times\mathbf{3}$ representations is given by (de Vega 1989 and references therein)

$$O_k^{3\times 3}=\sum_{rs} T_k^{rs}T_{k+1}^{rs}+\sinh\lambda\sum_{r,s}\mathrm{sgn}(r-s)T_k^{rr}T_{k+1}^{ss}+1 \tag{14}$$

where the 3×3 matrix T^{rs} has all its elements zero except the one in the r-row and s-column which is equal to one. This matrix has the eigenvalues

$$E_{\bar{3}}=1-\frac{(q+q^{-1})}{2} \qquad E_6=1+\frac{q+q^{-1}}{2} \tag{15}$$

which are different from those given by (13) (except for $q=1$). This raises the question of whether our $a=b=1$ Hamiltonian corresponds to a new SU(3) deformation. (We should keep in mind that the decomposition $\mathrm{SU}(3)\supset\mathrm{SO}(3)$ as opposed to $\mathrm{SU}(3)\supset\mathrm{SU}(2)\otimes\mathrm{U}(1)$ leaves only one element in the Cartan subalgebra instead of two, and thus the effect of the SO(3) deformation is unclear.) To answer this question, we have next computed the eigenvalues of the matrix

$$O_k(1,1;q)+\eta O_{k+1}(1,1;q) \tag{16}$$

where η is a parameter. If there is an SU(3) structure, the degeneracy of eigenvalues should be given by the branching rule

$$\mathbf{3}\times\mathbf{3}\times\mathbf{3}=\mathbf{1}+\mathbf{8}_1+\mathbf{8}_2+\mathbf{10}. \tag{17}$$

The actual calculation gives, for any value of η, six different eigenvalues instead of four: a singlet, two triplets, two quintets and a decuplet. So the single remnant of the SU(3) structure is the decuplet. It is thus probable that the higher degeneracies so far observed will disappear when we consider a higher number of sites. Moreover, as opposed to the $O_k^{3\times 3}$ defined by (14) which have the property that $O_k^{3\times 3}-1$ satisfy a Hecke algebra (Jimbo 1986), one can prove that there is no value of ξ for which

$$W_k=O_k(1,1;q)-\xi \tag{18}$$

satisfy the relation (4*b*) and thus the model is probably not integrable. The phase structure of this model remains to be studied. In particular, it would be interesting to know whether there is a second-order phase transition when q is a root of unity.

We now consider the case $a=0$, $b=1$. We notice again the relation $E_2(q)=E_1(q)$ as for $q=1$. We observe that the matrix

$$U_k=O_k(0,1;q)-1 \tag{19}$$

has the following matrix elements:

$$\langle m_1,m_2|U_k|m_1',m_2'\rangle=\delta_{m_1+m_2,0}\delta_{m_1'+m_2',0}q^{m_1+m_1'}(-1)^{m_1-m_1'} \qquad (-S\leqslant m\leqslant S,\ S=1) \tag{20}$$

and that the Temperley-Lieb algebra is satisfied with $\beta=[3]_q$. This result is interesting since by choosing $q=\mathrm{e}^{\mathrm{i}\phi}$ and $\phi\geqslant\pi/6$, one obtains $\beta\leqslant 2$ and presumably a second-order phase transition. One physical consequence of our result is that it includes a further integrable quantum spin chain which, via the Temperley-Lieb equivalence, should belong to the Potts universality class. What happens for $0<\phi<\pi/6$ and rational is an open question. A direct calculation of the $U_q[\mathrm{SU}(3)]$ invariant for the $3\times\bar{3}$ product gives

$$O_k^{3\times\bar{3}}=\sum_{r,s} T_k^{rs}T_{k+1}^{rs}+(q^{-2}-1)T_k^{22}T_{k+1}^{22}+(q^2-1)T_k^{33}T_{k+1}^{33}+(q^{-1}-1)(T_k^{12}T_{k+1}^{12}+T_k^{21}T_{k+1}^{21})$$
$$+(q-1)(T_k^{13}T_{k+1}^{13}+T_k^{31}T_{k+1}^{31}) \qquad (21)$$

and one can easily check that $O_k^{3\times\bar{3}}=O_k(0, 1; -q)$. We have found no explanation for the fact that the SU(3) structure is maintained in the $3\times\bar{3}$ case and is not in the 3×3 case.

A remarkable feature of $O_k^{3\times\bar{3}}$ is that it can be used to construct a Hamiltonian for a closed chain, which preserves translational invariance. (This is not possible for $O_k^{3\times3}$.) In the region of the second-order phase transition, one could determine by inspection of the finite-size scaling spectra the equivalent of the modular invariants with $U_q[\mathrm{SU}(3)]$ symmetry.

Equation (20) generalises to higher spins: for $S>1$, the matrices U_k again satisfy the Temperley-Lieb algebra with $\beta=[n]_q$, $n=2S+1$. This generalises a result obtained by Affleck (1989) and Batchelor and Barber (1989) for the undeformed case.

The condition $E_1(q)=E_0(q)$ which might allow the method of valence bond (Affleck *et al* 1988) to be applied is given by the line

$$\frac{a}{b}=(q+q^{-1})^2-1. \qquad (22)$$

Finally taking $a=-b=1$ in (12), one obtains the bulk terms of the Zamolodchikov and Fateev (1980) model. The effect of the surface terms will be considered elsewhere (Pasquier and Saleur 1989, Mezincescu *et al* 1989).

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